数理解析研究所講究録
740巻 1991年 303-314

Weakly Nonlinear Theory of Steady Hydrostatic Mountain Waves in a 2−layered Stratified Fluid of Infinite Depth over a 2−dimensional Mountain

By Motohki Ikawa (猪川 元興)

Meteorological Research Institute
1-1 Nagamine, Tsukuba, Ibaraki, 305 Japan

Abstract

Weakly nonlinear theory of steady hydrostatic mountain waves in 2-layered stratified Boussinesq fluid of infinite depth is presented. Weakly nonlinear effects (second-order correction) on drag, downslope wind and the steepening or flattening of the stream line are examined, and are found to be very sensitive to the depth of the lower layer, D, ℓ_2/ℓ_1 ($\ell_1 \equiv N_1/U$ and $\ell_1 \equiv N_1/U$; N_1 and N_2: Brunt-Vaisala frequencies of the lower layer and upper layer, respectively) and terrain shape. Drag obtained from linear theory is invariant under the change of π in $\ell_1 D$, while that obtained from weakly nonlinear theory is no more invariant under the change of π in $\ell_1 D$. The theory gives an estimate of the applicability range of linear theory. The theory is found to be in good agreement at least in a qualitative sense with nonlinear numerical solutions for some cases.

1. Introduction

Flows over orography have attracted meteorological interest for a long time. Linear aspects of flows under a radiative upper boundary condition over a 2-dimensional mountain have been made clear analytically. However, applicability of linear theory to severe downslope winds, foehns, blocking, clear air turbulence and so on is questionable because of the small amplitude assumption used in linear theory. Nonlinear aspects of flows have not been examined enough yet.

For a homogeneous atmosphere (constant horizontal velocity,U, and constant Brunt-Vaisala frequency,N), nonlinear steady aspects can be analytically examined by use of Long's equation (Long, 1953; Lilly and Klemp,1979). Lilly and Klemp's theory takes into account the nonlinear lower boundary and radiative upper boundary conditions, and their theory gives the critical inverse Froude number $\ell h \equiv Nh/U$ (h: the height of the mountain) for wave breaking and the magnitude of nonlinear enhancement of drag and downslope wind.

Nonlinear aspects of the flow of a multilayered fluid past a 2-dimensional mountain have not been examined enough yet, analytically or numerically. Durran(1986) and Ikawa(1990a) showed that, for some 2-layered fluids, the deviation of nonlinear numerical solutions from linear analytic solutions is larger even for a small inverse Froude number, ℓh, than that for homogeneous fluids. They found that, when $\ell_1 h$ becomes larger than a certain critical value, the flow becomes quite similar to the transitional flow familiar in hydraulic theory (Houghton and Kasahara, 1968), resulting in the high drag state with strong downslope wind. As far as $\ell_1 h$ is smaller than the critical value, the deviation from linear theory becomes gradually large as $\ell_1 h$

becomes large. No satisfactory nonlinear theory for it was presented in Durran and Ikawa.

Among recent theories which shed light on the transition of the flow into high drag state are those by Smith(1985), Mitsudera and Grimshaw(1989) (an extended one of Grimshaw and Smyth(1986), allowing for a radiative upper boundary condition) and Laprise and Peltier(1989). However, those theories are not directly applicable to the case of 2-layered stratified fluid studied by Durran and Ikawa because of restrictive assumptions made in those theories.

In this paper, weakly nonlinear (second-order perturbation) theory of steady hydrostatic mountain waves in 2-layered stratified fluid with a radiative upper boundary condition is presented. This theory cannot account for the catastrophic change of the flow resulting in high drag shown by Durran and Ikawa, implying that this is not a weakly but a strongly nonlinear phenomenon. However, the theory can account for the deviation of the flow from that of linear theory reported by Ikawa(1990a), such as the second-order enhancement of drag and downslope wind, as far as $\ell_1 h$ is small.

This paper is a simplified version of the original paper by Ikawa(1990b), where more detailed descriptions are given.

2. Formulation of weakly nonlinear theory

The 2-layered stratified fluid considered here has the constant horizontal mean wind Ui. In each layer, the Brunt-Vaisala frequency, N_j (j=1,2), is constant; The depth of the lower layer is D, and that of the upper layer is infinite. As an extension of Lilly and Klemp's equations (1979) for 1-layered stratified fluid of infinite depth, IN presented the exact equations for 2-layered stratified fluid of infinite depth, where nonlinear boundary conditions, nonlinear interface conditions and radiation conditions are taken into account. First, these equations are repeated below for reader's convenience.

Long's equation (1953) for each layer under the hydrostatic and Boussinesq approximation is written as follows:

$$(\partial^2/\partial z^2+\ell_j{}^2)\delta_j(x,z)=0 \;;\; \ell_j \equiv N_j/U_i \quad (\text{ for } j=1 \text{ and } 2), \qquad (1)$$

where $\delta_j(x,z)$ is the vertical displacement of an air parcel from its undisturbed height $\tilde{z}$ ($\delta_j=z-\tilde{z}$). The solution is written as:

for $z \leq Z_i(x)$:

$$\delta_1(x,z)=Zs(x)\cos(\ell_1(z-Zs(x)) - B_1(x)\sin(\ell_1(z-Zs(x)),$$

for $z \geq Z_i(x)$: (2)

$$\delta_2(x,z)=A_2(x)\cos(\ell_2(z-D-A_2(x))-B_2(x)\sin(\ell_2(z-D-A_2(x)),$$

where $Z_i(x)$ is the interface height given as

$$Z_i(x)=\delta_2(x,D+A_2(x))+D=A_2(x)+D. \qquad (3)$$

It is noted that the nonlinear lower boundary condition $\delta_1(x,Z_s(x))=Z_s(x)$ is automatically satisfied. The nonlinear interface conditions are

$$\delta_1(x,z)=\delta_2(x,z),$$

and

$$\partial\delta_1(x,z)/\partial z = \partial\delta_2(x,z)/\partial z$$

at the interface $(x,z)=(x,Z_i(x))$. This condition ensures the continuity of velocity ($u=U_i(1-\partial\delta/\partial z)$, $w=U_i\partial\delta/\partial x$) and pressure (Bernouli's equation: $1/2\rho_0 u^2+p+\rho gz$=constant on a streamline, where $\rho=\rho_0(1-N^2\tilde{z}/g)$) at the interface. They are rewritten as:

$$A_2(x)=Zs(x)\cos(\ell_1(D+A_2(x)-Zs(x)) - B_1(x)\sin(\ell_1(D+A_2(x)-Zs(x)) \tag{4}$$

$$\ell_2 B_2(x)=\ell_1\{Zs(x)\sin(\ell_1(D+A_2(x)-Zs(x)))+B_1(x)\cos(\ell_1(D+A_2(x)-Zs(x)))\}. \tag{5}$$

The upper radiation conditions (Lilly and Klemp,1979) require

$$B*_2(x)=-\mathrm{Hil}(A*_2), \qquad A*_2(x)=\mathrm{Hil}(B*_2), \tag{6}$$

where

$$A*_2(x)=A_2(x)\cos(\ell_2 A_2(x))+B_2(x)\sin(\ell_2 A_2(x)),$$
$$\tag{7}$$
$$B*_2(x)=-A_2(x)\sin(\ell_2 A_2(x))+B_2(x)\cos(\ell_2 A_2(x)).$$

Hil denotes the Hilbert transform defined as

$$\mathrm{Hil}(f)=\frac{1}{\pi}\ P\int\frac{f(x')}{x'-x}dx' , \tag{8}$$

for a function $f(x)$, where P denotes Cauchy's principal value.

For the three unknown functions $B_1(x)$,$A_2(x)$ and $B_2(x)$, three nonlinear relations (4),(5) and (6) exist. However, these equations are transcendental equations and difficult to be solved. As pointed out by IN, Eqs.(4), (5) and (6) yield Smith's(1985) equation for the special cases of $\ell_2=0$.

Next, to solve the above transcendental equations approximately, the functions B_1 , A_2 and B_2 are assumed to be expanded in powers of the small parameter $\varepsilon\equiv\ell_1 h$ as follows:

$$B_1=h(B_{10}+\varepsilon B_{11}+O(\varepsilon^2)), \tag{9-1}$$

$A_2 = h(A_{20} + \varepsilon A_{21} + O(\varepsilon^2))$, (9-2)

$B_2 = h(B_{20} + \varepsilon B_{21} + O(\varepsilon^2))$, (9-3)

where h is a representative height of the mountain, $Z_s(x)$. The normalized mountain shape function Z_{s0} is given as

$Z_{s0}(x) \equiv Zs(x)/h$. (10)

Substituting the above relations into Eqs.(4)-(5) and assuming

$$\varepsilon^{-1} \gg \alpha \equiv \ell_2/\ell_1 \sim 1 \gg \varepsilon, \tag{11}$$

the first- and second-order solutions are obtained as below:

$B_{10} = -(sZ_{s0}(x) + r\mathrm{Hil}(Z_{s0}(x)))$, (24)

$A_{20} = (\cos\Delta + s\sin\Delta)Z_{s0} + r\sin(\Delta)\mathrm{Hil}(Z_{s0}(x))$, (25)

$B_{20} = -(\cos\Delta + s\sin\Delta)\mathrm{Hil}(Z_{s0}) + r\sin(\Delta)Z_{s0}(x)$, (26)

where

$$r \equiv \frac{2\alpha}{1+\alpha^2+(1-\alpha^2)\cos(2\Delta)}, \tag{27}$$

$$s \equiv \frac{(1-\alpha^2)\sin(2\Delta)}{1+\alpha^2+(1-\alpha^2)\cos(2\Delta)}. \tag{28}$$

B_{11} is given as below*:

$$B_{11} = \frac{1}{\sin^2\Delta + (\cos\Delta/\alpha)^2}\left[(\cos\Delta/\alpha)(-\mathrm{Hil}(C_1)+C_3-C_2) - \sin\Delta(\mathrm{Hil}(C_2)+C_4-C_1)\right]$$

$$= B_{11}^{zz}Z_{s0}^2 + B_{11}^{zh}Z_{s0}\mathrm{Hil}(Z_{s0}) + B_{11}^{hh}\mathrm{Hil}(Z_{s0})^2$$

$$+ HB_{11}^{zz}\mathrm{Hil}(Z_{s0}^2) + HB_{11}^{zh}\mathrm{Hil}(Z_{s0}\mathrm{Hil}(Z_{s0})) + HB_{11}^{hh}\mathrm{Hil}(\mathrm{Hil}(Z_{s0})^2), \tag{29}$$

where

$$B_{11}{}^{zz} \equiv \{\cos\Delta(\cos\Delta+s\sin\Delta)^2 -(1/\alpha^2)\cos\Delta(\cos\Delta+s\sin\Delta-1)(\cos\Delta+s\sin\Delta)$$
$$+\alpha r\sin^2\Delta(\cos\Delta+s\sin\Delta)-\sin\Delta(\cos\Delta+s\sin\Delta-1)(\sin\Delta-s\cos\Delta)\}$$
$$/[\sin^2\Delta+(\cos\Delta/\alpha)^2]. \tag{30}$$

For the definitions of $B_{11}{}^{zh}$, $B_{11}{}^{hh}$, $HB_{11}{}^{zz}$, $HB_{11}{}^{zh}$, $HB_{11}{}^{hh}$, see Ikawa(1990b).

Once B_{11} is obtained, A_{21} and B_{21} are readily obtained from Eqs. (15) and (16). A_{21} and B_{21} are also expressed in linear combination of the six functions of x, i.e., $Z_{s0}{}^2$, $Z_{s0}\mathrm{Hil}(Z_{s0})$, $\mathrm{Hil}^2(Z_{s0})$... $\mathrm{Hil}(\mathrm{Hil}^2(Z_{s0}))$.

Surface pressure p_s is given (see Eq.(31) of Lilly-Klemp) as

$$p_s=-\rho/2(U^2-U_i{}^2+N^2Z_s(x)^2)$$

$$=-\frac{\rho U_i{}^2}{2}\left[2\varepsilon B_{10}+\varepsilon^2(B_{10}{}^2+2B_{11})+O(\varepsilon^3)+\varepsilon^2 Z_{s0}{}^2(x)\right].$$

Neglecting the $O(\varepsilon^3)$ terms in p_s, wave drag is given as

$$DRG\equiv\int p_s\frac{dZs}{dx}dx=-\frac{\rho N_1U_ih^2}{2}\int[2B_{10}+\varepsilon(B_{10}{}^2+2B_{11})]\frac{dZ_{s0}}{dx}dx$$

$$=-\frac{\rho N_1U_ih^2}{2}\{-2sD(Z_{s0})-2rD(\mathrm{Hil}(Z_{s0}))$$
$$+\varepsilon[(2B_{11}{}^{zz}+s^2)D(Z_{s0}{}^2)$$
$$+(2B_{11}{}^{zh}+2sr)D(Z_{s0}\mathrm{Hil}(Z_{s0}))+(2B_{11}{}^{hh}+r^2)D(\mathrm{Hil}^2(Z_{s0}))$$
$$+2HB_{11}{}^{zz}D(\mathrm{Hil}(Z_{s0}{}^2))+2HB_{11}{}^{zh}D(\mathrm{Hil}(Z_{s0}\mathrm{Hil}(Z_{s0})))$$
$$+2HB_{11}{}^{hh}D(\mathrm{Hil}(\mathrm{Hil}^2(Z_{s0})))]\}, \tag{37}$$

where the operator D on a function f(x) is defined as

$$D(f) = \int f \frac{dZ_{s0}}{dx} dx. \tag{38}$$

3. Weakly nonlinear effects for the case of a bell-shaped mountain and comparison with nonlinear numerical solutions

The formula derived in section 2 are applied to the case of a bell-shaped mountain defined as

$$Z_s = \frac{ha^2}{x^2 + a^2}. \tag{39}$$

The functions and integrals necessary in computing flow patterns and drag are listed in Table 1 of Ikawa(1990b) and shown in Fig.1 of Ikawa(1990b). From Eq.(37) and Table 1 of Ikawa(1990b), surface pressure drag for a bell shaped mountain is obtained as

$$DRGN \equiv DRG/DRGL = [r - \frac{\varepsilon}{2}(sr + B_{11}{}^{zh} + 2HB_{11}{}^{zz})], \tag{40}$$

where DRGL is the drag by linear theory for the case of $\ell_2/\ell_1 = 1$ defined as

$$DRGL \equiv \frac{\pi \rho N_1 U_i h^2}{4}. \tag{41}$$

To check the validity of the theory, the dependency of DRGN on $\ell_1 h$ obtained from the weakly nonlinear theory is compared with that of nonlinear numerical solutions for the case of $\ell_2/\ell_1 = 0.4$ and a bell shaped mountain (Eq.(39) with $a\ell_1 = 45$). Specifications for numerical experiments are the same in Ikawa

(1990a); the vertical grid distance Δz=200m ($\ell_1 \Delta z \doteqdot 0.1\pi$) is used; the lateral boundary condition is the same as CS (cyclic and small) in Ikawa. Some numerical data are the same in Ikawa, and some are newly computed for the present purpose.

In Fig.1, the comparison between the two for the three cases of $\ell_1 D=0.72\pi$, 1.49π and 1.39π is shown. Weakly nonlinear theory predicts the large increase in DRGN for $\ell_1 D=0.72\pi$ and the large and slight decreases for $\ell_1 D=1.39\pi$ and 1.49π, as $\ell_1 h$ becomes large. As shown in Fig.1, these features are also seen in numerical counterparts, but quantitative agreement is not good. For the case of $\ell_1 D=0.72\pi$, the agreement between the two appears to be very good up to $\ell_1 h=0.375$; however, as mentioned in Ikawa, the shock-like disturbance propagating downstream is seen in the numerical solution with $\ell_1 h=0.375$, and the flow pattern and its time dependency differs between the two.

One of the reasons might be the neglect of the higher-order terms and nonhydrostatic effects in weakly nonlinear theory. Major reason is probably the numerical errors in nonlinear time-dependent numerical solutions associated with nonsteadiness of the numerical solutions, insufficient boundary conditions and finite discretization errors. DRGN is very sensitive to $\ell_1 D$; for example, near $\ell_1 D=1.4\pi$, the difference of 0.1 in $\ell_1 D$ yields the difference of more than 30% in DRGN as shown in Fig.4 of Ikawa(1990b). The use of the vertical grid distance $\ell_1 \Delta z \doteqdot 0.1\pi$ may result in large error of the effective $\ell_1 D$ in the finite difference numerical model. To see the effects of the finite discretization error in representing the interface on the numerical mesh, the experiment with the finer vertical resolution (Δz=100m instead of Δz=200m) is conducted for the case of

$\ell_1 D=1.49\pi$, and the result is indicated by the symbol Δ in Fig.1b. The finer resolution yields a closer value of DRGN to that obtained by weakly nonlinear theory. Durran(1986) also reported large discrepancy as much as 25% in DRGN between the linear and nonlinear numerical solutions for the 2-layered fluid with $\ell_1 h=0.001$.

Taking into account these situations, it might be said that the theory is in fairly good agreement at least in a qualitative sense with nonlinear numerical solutions. The other cases reported by Ikawa appear to be not inconsistent with the results by the weakly nonlinear theory.

References

Durran, D.R., 1986: Another look at downslope windstorms. Part 1. J.Atmos.Sci., 43, 2527-2543.

Ikawa, M. and Y.Nagasawa, 1989: A numerical study of a dynamically induced foehn observed in the Abashiri-Ohmu area. J.Meteor.Soc.Japan, 67, 429-458.

Ikawa, M., 1990a: High drag states and foehns of a 2-layered stratified fluid past a 2-dimensional mountain. J.Meteor.Soc.Japan, 68, 163-182.

Ikawa, M., 1990b: Weakly nonlinear aspects of steady hydrostatic mountain waves in a 2-layered stratified fluid over a 2-dimensional mountain. J.Meteor.Soc.Japan,68, July (in press)

Laprise R. and W.R.Peltier, 1989: The linear stability of nonlinear mountain waves: Implications for the understanding of severe downslope windstorms. J.Atmos.Sci., 46, 545-564.

Lilly, D.K. and J.B.Klemp, 1979: The effects of terrain shape on non-linear hydrostatic mountain waves. J.Fluid Mech.,95, 241-261.

Long,R.R., 1953: Some aspects of the flow of stratified fluids. I. A theoretical investigation. Tellus, 5, 42-58.

Mitsudera H. and R.Grimshaw, 1989: Effects of radiative damping on resonantly generated internal gravity waves. Applied Mathematics Preprint, School of Mathematics, The University of New South Wales (Australia).

Smith, R.B., 1985: On severe downslope winds. J.Atmos.Sci., 42, 2597-2603.

Fig.1 a) Comparison of the normalized drag (DRGN) obtained from weakly nonlinear theory (drawn by the dotted line) with those obtained from nonlinear numerical solutions (indicated by the symbol ×) for the case of $\ell_2/\ell_1=0.4$ and $\ell_1 D=0.72\pi$. The ordinate denotes DRGN and the abscissa denotes $\varepsilon\equiv\ell_1 h$.

b) the same as a) but for $\ell_1 D=1.49\pi$. For the explanation of the symbol Δ in Fig.1b, see text.

c) the same as a) but for $\ell_1 D=1.39\pi$. The lower and upper dotted lines show DRGNs obtained from weakly nonlinear theory for $\ell_1 D=1.38\pi$ and $\ell_1 D=1.45\pi$, respectively.